ISOMORPHISMS BETWEEN FUCHSIAN GROUPS

ALBERT MARDEN*

INTRODUCTION.

Let G be a non-cyclic fuchsian group acting in the unit disk Δ. Then in general Δ is branched over a nowhere dense set of points $b(G)$ in the Riemann surface $S(G) = \Delta/G$. If $\gamma \subset S(G) - b(G)$ is a loop with origin p and $p^* \in \Delta$ lies over p, the lift γ^* of γ from p^* terminates at $T(p^*)$ for some $T \in G$; T is said to be <u>determined</u> by γ. Two transformations determined by γ are conjugate in G. A <u>boundary transformation</u> is an element of G which is determined by a simple loop in $S(G) - b(G)$ which is retractable in $S(G) - b(G)$ to an ideal boundary component of $S(G)$. Thus a boundary transformation is either hyperbolic or parabolic (since G is not finite cyclic) and every parabolic transformation is the power of a parabolic boundary transformation (e.g. [7]). In addition if a boundary transformation $A \in G$ satisfies $A = B^m$ for some $B \in G$ then $m = \pm 1$ (see [6]).

The group G is finitely generated if and only if $b(G)$ is a finite set and $S(G)$ is the interior of a finitely punctured compact bordered surface (e.g. [5]).

A homeomorphism $f^*: \Delta \longrightarrow \Delta$ is said to <u>induce</u> an isomorphism $\Phi: G \longrightarrow H$ if $\Phi(A) = f^*Af^{*-1}$ for all $A \in G$. If this is the case then f^* projects to a homeomorphism $f: S(G) \longrightarrow S(H)$ which sends $b(G)$ to $b(H)$. Alternatively a homeomorphism $f: S(G) - b(G) \longrightarrow S(H) - b(H)$ is said to <u>induce</u> Φ if there is an extension $f: S(G) \longrightarrow S(H)$ and a lift $f^*: \Delta \longrightarrow \Delta$ which induces Φ. Other choices of lifts will induce isomorphisms of the form $A \longmapsto T_2\Phi(T_1AT_1^{-1})T_2^{-1}$ for $T_1 \in G$, $T_2 \in H$. Obviously such homeomorphisms f and f^*

---

*Supported in part by the National Science Foundation.

preserve boundary transformations. The main purpose of this paper is to present a proof of the converse.

THEOREM. *Suppose* G *and* H *are finitely generated fuchsian groups and* $\Phi: G \to H$ *is an isomorphism with the property that when* $\Phi(A) \in H$ *is a boundary transformation,* A *is too. Then there exists a homeomorphism* $f: S(G) - b(G) \to S(H) - b(H)$ *which induces* Φ. *Furthermore,* Φ *uniquely determines the homotopy class of* f *in* $S(G) - b(G)$.

We remark that the theorem can also be presented with the apparently less stringent but actually equivalent hypothesis that when $\Phi(A)$ is a boundary transformation, A is a power of one.

The first statement is due to Fenchel-Nielsen and is contained in their famous manuscript [2] which, although unpublished, has had great influence on the theory of fuchsian groups. Their approach to the theorem is by means of their intersecting axes theorem (which we prove here as Proposition 4.1) and the same path is followed here. This latter result gives a simple condition in terms of the behavior of the axes under Φ that implies Φ is induced by a homeomorphism. It is important in its own right since it applies to arbitrary fuchsian groups, not just finitely generated ones.

Here the intersecting axes theorem is applied in the following way. A suitable collection of axes of G gives a geometrically meaningful tesselation of Δ; isomorphic groups give equivalent tesselations. This allows us to extend to all of Δ a map defined first on these axes of G. However the first step in the proof, and perhaps the more interesting, is to get from the hypothesis of the theorem to the hypothesis of the intersecting axes theorem. This is done by using Φ to set up a bijection φ between the lattice points of G and H and then, using the manner of their accumulation to $\partial\Delta$, to show the extension of φ to the fixed points of G on $\partial\Delta$ preserves the order in which they are located on $\partial\Delta$.

The second statement of the theorem follows from the fact, proved by different methods in [1] and [6], that a homeomorphism of $S(G) - b(G)$ onto itself which induces the identity automorphism of G is homotopic in $S(G) - b(G)$ to the identity.

The theorem originates with the classical result of Nielsen that every automorphism of the fundamental group of a closed surface is induced (up to an inner automorphism) by a homeomorphism of the surface. Nielsen's proof was given in terms of fuchsian groups but with the machinery now available in topology, one can give short proofs of this and more general results (cf. [11, Lemma 1.4.3]). Beyond this classical case the first results that appeared in the literature were those of Zieschang [12,13] (see also the monograph [14]). He gave a purely algebraic development of the subject including proofs for the case that $S(G)$ is compact and for the case $b(G) = \phi$; he has reported that these methods also yield a proof of the first statement of the theorem in general (personal communication). Using extremal properties of Teichmüller mappings, Macbeath [4] showed how Nielsen's result can be used to give a short elegant proof when $S(G)$ is compact (but $b(G) \neq \phi$) and the method has more general applicability as well. Tukia [10] rediscovered the significance of the intersecting axes property and gave a proof of this result that covers most cases, pointing out how it implies the theorem when $S(G)$ is compact. Much of the work cited also contains a discussion of the case that G contains reflections (in particular the Fenchel-Nielsen proof includes this case), a situation we ignore completely here.

This is a revision of my proof distributed in preprint form in 1970 and I want to thank L. Greenberg for his suggestions and interest in this work spanning eight years. Above all it should not be forgotten that the proof presented here is in the spirit of Fenchel-Nielsen and merely reflects the importance of that work.

1. GEODESICS AND LOOPS

1.1. Let γ be a simple loop in $S(G) - b(G)$ with origin p and fix $p^* \in \Delta$ over p. The lift γ_0^* of γ from p determines an element T_γ of G which is parabolic (resp. elliptic) if and only if γ is retractable in $S(G) - b(G)$ to a puncture of $S(G)$ (resp. a point of $b(G)$) [5,6]. If T_γ is hyperbolic, the open Jordan arc $\gamma^* = \cup_{n=-\infty}^{\infty} T_\gamma^n(\gamma_0^*)$ is called the <u>lift-chain</u> of γ determined by γ_0^* or by T_γ. It has a natural orientation induced from γ which directs it from the repulsive ζ_r toward the attractive fixed point ζ_a of T_γ.

Suppose $T = T_\gamma$, determined by the simple loop γ, is hyperbolic. The non-euclidean line $\alpha^*(T)$ directed from ζ_r toward ζ_a is called the <u>axis</u> of T. Set $\alpha(\gamma) \equiv \alpha(T) = \pi(\alpha^*(T))$ where π is the natural projection $\Delta \longrightarrow S(G)$. If γ_1, γ_2 are simple loops the following elementary facts are clear from non-euclidean geometry.

(*i*) If $\gamma_1 \cap \gamma_2 = \phi$ then $\alpha(\gamma_1) \cap \alpha(\gamma_2) = \phi$.

(*ii*) If γ_1 crosses γ_2 exactly once so does $\alpha(\gamma_1)$ and $\alpha(\gamma_2)$.

From [6] we see that the following holds: If γ is a simple loop, $\alpha(\gamma)$ is with one exception a simple loop in $S(G) - b(G)$ which is freely homotopic in $S(G) - b(G)$ to γ. $\alpha(\gamma)$ is a geodesic in the singular (if $b(G) \neq \phi$) metric on $S(G)$ induced by π from the Poincaré metric on Δ. The single exception occurs when γ bounds a simply connected region in $S(G)$ containing exactly two points $x_1, x_2 \in b(G)$ and these are both of order two. Then $\alpha(\gamma)$ is an arc from x_1 to x_2.

More generally let $\gamma \subset S(G) - b(G)$ be any loop from p which determines a hyperbolic transformation T_γ. Then γ can be replaced by another loop δ whose lift δ_0^* also determines T_γ but in addition δ_0^* and the resulting lift-chain δ^* are Jordan arcs. δ is a simple loop if and only if $S(\delta^*) \cap \delta^* = \phi$ for all $S \in G$ not a power of T_γ.

Suppose $T \in G$ is hyperbolic with axis $\alpha^*(T)$. Then $T = T_\gamma$ for a simple loop $\gamma \subset S(G) - b(G)$ if and only if $S\alpha^*(T) \cap \alpha^*(T) = \phi$ for all $S \in G$ which is not a power of T, with the exception of the case cited above.

2. THE LATTICE CONSTRUCTION

2.1. Primarily in order to simplify the proof that if T is a boundary transformation then $\Phi(T)$ is one as well we are going to assume in this chapter that neither G nor H contains parabolic transformations. Because of the following reasoning, this is no restriction. If for example an ideal boundary component x of $S(G)$ is a puncture, remove a small neighborhood $D \subset S(G) - b(G)$ of x. Once this is done for all punctures of $S(G)$ we can represent the smaller surface $S(G) - \cup D$ but with the same branch set $b(G)$ by a new group G'. There is a natural isomorphism $\varphi_1\colon G' \longrightarrow G$ that arises from the embedding of $S(G')$ into $S(G)$. Clearly T is a boundary transformation of G if and only if $\varphi_1(T)$ is one for G'. Denote by H' the corresponding group for H and by $\varphi_2\colon H' \longrightarrow H$ the corresponding isomorphism. We see that Φ is induced by a homeomorphism $S(G) \longrightarrow S(H)$ if and only if $\varphi_2 \Phi \varphi_1^{-1}$ is induced by one sending $S(G')$ to $S(H')$. So replace G by G', H by H', Φ by $\varphi_2 \Phi \varphi_1^{-1}$ and then revert to the original notation.

Now we can characterize boundary transformations of G (and correspondingly of H) by the condition that $T \in G$ is a boundary transformation if and only if $\alpha^*(S) \cap \alpha^*(T) = \phi$ for all $S \in G$ not a power of T.

2.2. Fix a point $0 \in \Delta$ which is not a fixed point of G or $H = \Phi(G)$ as base point. A <u>topological fundamental region</u> $\mathcal{G}$ for G based at 0 is contructed as follows. In $S(G) - b(G)$ draw $2g$ (g = genus $S(G)$) simple loops (α_i, β_i), $1 \leq i \leq g$, from $\pi(0)$, mutually disjoint except at 0, so that $\Pi\ \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$ is the oriented relative boundary of a planar subregion S' of $S(G) - b(G)$. In S' draw simple arcs $\{\gamma_i\}$ from $\pi(0)$ to each ideal boundary component (using the natural compactification) of $S(G)$ and to each point in $b(G)$ which are mutually disjoint except at $\pi(0)$. The totality of $\{\alpha_i, \beta_i, \gamma_i\}$ cuts $S(G) - b(G)$ into a simply connected region R_1. Let $\mathcal{G}$ be the closure in $\bar{\Delta}$ of one of the lifts of R_1 which contains 0.

The totality of images of 0 under G are called <u>vertices</u> of G. A side of $\mathcal{G}$ either joins two vertices or joins a vertex to an elliptic fixed point or to a point on $\partial\Delta$. It is convenient to use the term <u>ray</u> of $\mathcal{G}$ for either a) a side of $\mathcal{G}$ which joins two vertices, or b) the union of two sides σ_1, σ_2 of $\mathcal{G}$ where $\sigma_1 \cup \sigma_2$ joins two vertices of $\mathcal{G}$ via an elliptic fixed point of G or via an arc of $\partial\Delta$ which is included in the ray. The totality of images of rays of $\mathcal{G}$ under G will be referred to as <u>rays of</u> G. If two rays γ_1, γ_2 of G have a vertex in common but $\gamma_1 \cup \gamma_2$ is not a Jordan arc, it will be one (or it will be a point) after cancellation of a common segment (or segments). In writing a union of rays it will always be assumed all possible cancellations are carried out.

If $\mathcal{G}$ contains n vertices then there are n rays of G which have 0 as an end point; denote the other end points by $S_i(0)$, $i = 1,\ldots,n$, $S_i \in G$. The transformations S_i generate G. The points $S_i(0)$ are said to be <u>adjacent</u> to 0. More generally if $P = S(0)$ is a vertex, the points $SS_iS^{-1}(P)$ are said to be <u>adjacent</u> to P (P adjacent to Q implies Q adjacent to P). A <u>path</u> in G from a vertex P to a vertex Q is a finite set of vertices

$P = P_0, P_1, \ldots, P_k = Q$ such that P_i is adjacent to P_{i-1}. The <u>length</u> of the path is k. It is clear by reference to the network of rays that there is a path between any two vertices. If $S \in G$, then $S(P_0), \ldots, S(P_k)$ is a path from $S(P)$ to $S(Q)$.

<u>Property 1</u>. Given $\zeta \in \partial\Delta$, set

$$N_r(\zeta) = \bar{\Delta} \cap \{|z-\zeta| \le r\}.$$

If ζ is a limit point there exists $s < r$ such that any two vertices in $N_s(\zeta)$ can be joined by a path in $N_r(\zeta)$.

Indeed, choose s so small that any fundamental region $S(G)$, $S \in G$, which meets $N_s(\zeta)$ does not meet the relative boundary of $N_r(\zeta)$ in Δ. Then if P, Q are vertices in $N_s(\zeta)$ draw a line segment σ in $N_s(\zeta) \cap \Delta$ from P to Q; a path from P to Q can be constructed from the sides of those $S(G)$ which meet σ.

2.3. The isomorphism Φ determines a $1-1$ map φ of the set of vertices $\{G(0)\}$ onto the set of vertices $\{H(0)\}$ of H:

$$\varphi: S(0) \longrightarrow \Phi(S)(0), \qquad S \in G.$$

If P is a vertex of G and $S_1 \in G$ then

$$\varphi(S_1(P)) = \Phi(S_1)(\varphi(P)).$$

We define adjacent vertices of H as the <u>image under</u> φ of adjacent vertices of G and paths in H as the <u>image under</u> φ of paths in G. Let $\mathcal{H}$ be a topological fundamental region for H constructed with respect to 0.

<u>Property 2</u>. Let ζ be a limit point of H. Given $r > 0$ there exists $0 < s < r$ such that any pair of vertices of H in $N_s(\zeta)$ can be joined by a path in $N_r(\zeta)$.

For choose a path in H between each pair P, Q of vertices of H which are also vertices of $\mathcal{H}$ (namely the image of a path in G from $\varphi^{-1}(P)$ to $\varphi^{-1}(Q)$) and let M be the maximum length of these paths. Then choose $r_1 < r$ so small that for any vertex P of H in $N_{r_1}(\zeta)$ all paths of length $\leq M$ from P are contained in $N_r(\zeta)$. Finally pick $s < r_1$ so that for each pair P, Q of vertices of H in $N_s(\zeta)$, there is a sequence $P_0 = P, P_1, \ldots, P_k = Q$ of vertices in $N_{r_1}(\zeta)$ such that P_i and P_{i-1} are vertices of the fundamental region $T_i(\mathcal{H})$, $T_i \in H$. Consequently P_{i-1} and P_i for each i, and hence P and Q, can be joined by a path in $N_r(\zeta)$.

2.4. Extend the function φ to the fixed points of G on $\partial\Delta$ by specifying that the fixed points of $S \in G$ are to correspond to those of $\Phi(S)$, attractive corresponding to attractive, repulsive to repulsive. Obviously this could be done for the elliptic fixed points too but we don't need to do so.

A loop γ, not necessarily a simple loop, from $\pi(0)$ in $S(G) - b(G)$ can be replaced by an expression in terms of a given set of generators of $\pi_1(S(G) - b(G))$ which also determines $T = T_\gamma$. Making use of this one finds a sequence of vertices $P_0, P_1, \ldots, P_n = T(P_0)$ of G such that (*i*) P_i is adjacent to P_{i-1} for all i, and (*ii*) if $\gamma_0^* = \cup\, \gamma_i^*$ where γ_i^* is the ray from P_{i-1} to P_i then $\cup_{-\infty}^{\infty} T^k(\gamma_0^*)$ with the two fixed points of T adjoined after cancellation of common segments and elimination of internal loops is a closed Jordan arc γ^* in the closed disk $\overline{\Delta}$.

Let $\overline{\Delta}_L$ (resp. $\overline{\Delta}_R$) be the union of those components of $\overline{\Delta} - \gamma^*$ which lie to the left (resp. right) of γ^* (there may be more than one component because interior points of γ^* may lie on $\partial\Delta$). Denote by $L(T)$ (resp. $R(T)$) the corresponding set of vertices and hyperbolic fixed points of G which lie in $\overline{\Delta}_L \cup \gamma^*$ (resp. $\overline{\Delta}_R \cup \gamma^*$) and set $\Gamma(T) = L(T) \cap R(T)$, the set of vertices and hyperbolic fixed points lying on γ^*. Then $L = L(T)$ (and $R(T)$) have the following properties.

(*i*) Two vertices in L can be joined by a path in L.

(*ii*) A path joining $P \in L$ to $Q \notin L$ contains a vertex on Γ.

(*iii*) L contains the fixed points (resp. the attractive fixed point) of a non-elliptic $S \in G$ if and only if $S^k(P) \in L$ for all large $|k|$ (resp. all large k) and any given vertex $P \in L$. (If S is not a power of $T = T_\gamma$ we can take P as any vertex of G).

(*iv*) If L contains one fixed point of a boundary transformation, it contains the other as well.

(*v*) T is a power of a boundary transformation if and only if one of L, R is equal to Γ. In other cases both $\bar{\Delta}_L$ and $\bar{\Delta}_R$ contain limit points of G.

These properties are easily deduced by thinking of $\bar{\Delta}_L$ as a union of fundamental regions $T_i(G)$, $T_i \in G$, with boundary Γ.

2.5. The set of points in $\Gamma' = \varphi(\Gamma)$ is invariant under and accumulates only to the fixed points of $\Phi(T) \in H$. Indeed, Γ' is the closure of the orbit under $\Phi(T)$ of the finite set $\varphi(\gamma_0^* \cap \Gamma)$.

Set $L' = \varphi(L)$, $R' = \varphi(R)$. We have to show that, modulo orientation, L', R' play the same role in H as L, R do in G.

<u>Property 3</u>. Suppose $\{P_k\}$ is a sequence of vertices of H in L' such that $\lim P_k = \zeta \in \partial\Delta$ where $\zeta \notin \Gamma'$. Then there exists $s > 0$ such that all vertices and hyperbolic fixed points of H which are in $N_s(\zeta)$ are also in L'.

To prove this choose r so small that $N_r(\zeta)$ does not contain a vertex on Γ'. By Property 2 there exists $s < r$ such that any two vertices in $N_s(\zeta)$ can be joined by a path in $N_r(\zeta)$. For all large k, $P_k \in N_s(\zeta)$. Suppose first that Q is a vertex in $N_s(\zeta)$ but $Q \notin L'$. Join P_k to Q by a path in $N_r(\zeta)$. But by property *(ii)* of L (§2.4), this path contains a vertex on Γ', a contradiction. If now $x \in N_s(\zeta)$ is say the attractive fixed point of $S \in H$ then for any vertex Q of H, $S^k(Q) \in N_s(\zeta)$ for all large k. Therefore $S^k(Q) \in L'$ and hence $\Phi^{-1}(S)^k(\varphi^{-1}(Q)) \in L$ for all large k. This implies the attractive fixed point of $\Phi^{-1}(S)$ is in L and hence $x \in L'$.

Property 3 implies the following.

<u>Property 3'</u>. Suppose for some $W \in H$ and $P \in L'$, $W^k(P) \in L'$ for all large k. Then the attractive fixed point of W lies in L'.

2.6. Let Σ' be the set of points $\zeta \in \partial\Delta$ which satisfy either
(a) $\lim P_k = \zeta$ for a sequence $\{P_k\}$ of vertices in L', or
(b) there is a boundary transformation $B \in H$ with fixed points in L' and ζ lies in that closed interval on $\partial\Delta$ bounded by the fixed points of B yet containing no other limit points of H. It is here, for (b), that we are making essential use of the fact that if B is a boundary transformation then $\Phi^{-1}(B)$ is one. For then not just one but both fixed points of $\Phi^{-1}(B)$ are in L (property *(iv)* of L) and hence both fixed points of B are in L'.

<u>Property 4</u>. If T_γ is not a power of a boundary transformation then Σ' is a closed interval on $\partial\Delta$ bounded by the fixed points x_1, x_2 of $\Phi(T_\gamma)$.

Indeed Σ' is a closed set on $\partial\Delta$. For if ζ lies in its closure there is a sequence $\{\zeta_n\} \in \Sigma'$ with $\lim \zeta_n = \zeta$. If $\zeta \notin \Sigma'$ then ζ must be a limit point of H and the ζ_n can be taken to be limit points as well. Thus we can find a sequence of vertices $\{P_k\}$ in L' with $\lim P_k = \zeta$, a contradiction. On the other hand Property 3 and the requirement (b) in the definition of Σ' imply that $\Sigma' - \{x_1,x_2\}$ is an open set. So either Σ' is one of the closed intervals bounded by x_1 and x_2, or $\Sigma' = \partial\Delta$. At this point we use the hypothesis that T_γ is not the power of a boundary transformation. For then R' contains limit points of $H \neq \{x_1,x_2\}$ so Σ' is an interval.

As a consequence of Property 4 we can identify R' and L' with the complementary closed intervals of $\partial\Delta$ determined by the two fixed points of $\Phi(T)$. The corresponding situation with respect to G is that L and R can be identified with intervals on $\partial\Delta$ that lie to the left and right respectively of the axis $\alpha^*(T)$ directed toward the attractive fixed point of T. L' corresponds to L in the sense that the fixed points (attractive fixed points) of $S \in G$ lies in L if and only if the fixed points (resp. attractive fixed point) of $\Phi(S)$ lies in L'. In particular we have:

<u>Property 4'</u>. Suppose $T = T_\gamma$ is not a power of a boundary transformation. Then the axis of $S \in G$ crosses that of T if and only if the axis of $\Phi(S)$ crosses that of $\Phi(T)$.

2.7. <u>Property 5</u>. If $A \in G$ is a boundary transformation so is $\Phi(A)$.

For suppose $\Phi(A)$ is not (and hence not a power of one either). Then if the axis $\alpha^*(V)$, $V \in H$, crosses $\alpha^*(\Phi(A))$ it follows from Property 4' that $\Phi^{-1}(V)$ must be a power of a boundary transformation. Before finding such a V observe that H contains boundary transformations. For otherwise H would have a torsion free, normal subgroup H_0 of finite index, isomorphic to the fundamental group of a closed surface. The corresponding group $G_0 = \Phi^{-1}(H_0)$ would give rise to a compact covering surface, contradicting the fact that no finite sheeted covering of $S(G)$ is compact. Consequently we can choose boundary transformations V_1, V_2 of H so that the fixed points of V_1 lie to the left of $\alpha^*(\Phi(A))$ while those of V_2 lie to the right. Using an argument of Tukia [10], for sufficiently large n, the elements $V^+ = V_1^n V_2^n$ and $V^- = V_1^n V_2^{-n}$ map the exterior of a small circle about a fixed point of V_2 onto the interior of a small circle about a fixed point of V_1. That is, we can find (arbitrarily large) n so that $\alpha^*(V^+)$ and $\alpha^*(V^-)$ cross $\alpha^*(\Phi(A))$. Hence $\Phi^{-1}(V^+) = U_1^n U_2^n$ and $\Phi^{-1}(V^-) = U_1^n U_2^{-n}$ are powers of boundary transformations and the $U_i = \Phi^{-1}(V_i)$ are already known to be. But this is impossible: For let G_0 be the group generated by U_1, U_2. G_0 is a free group, and the surface Δ/G_0 is a 3-holed sphere. Either U_1U_2 or $U_1U_2^{-1}$ is a boundary element in G_0. Replacing U_2 by U_2^{-1}, if necessary, we may suppose that U_1U_2 is a boundary element. Then the boundary elements in G_0 are U_1, U_2, U_1U_2 and their conjugates. By referring to the abelianized group, it is clear that $U_1^n U_2^{-n}$ is not a boundary element or a power of one.

<u>Property 5'</u>. Properties 4 and 4' hold for any $T = T_\gamma$.

2.8. We will call Φ <u>orientation preserving</u> if the interval on $\partial\Delta$ determined by $L'(T)$ lies to the left of the axis of $\Phi(T_\gamma)$ directed toward its attractive fixed point, otherwise Φ is called orientation reversing. Note that the question of whether Φ is orientation preserving is completely determined once one knows the location of the attractive fixed point of $\Phi(S)$ for some $S \in G$ whose axis crosses that of T. Our definition makes sense because of:

<u>Property 6</u>. If Φ is orientation preserving with respect to one choice of γ and T_γ it is orientation preserving with respect to

every other choice.

To prove this orient all axes in the direction of the attractive fixed point. Suppose $T = T_\gamma$ and $U = T_\delta$ are two choices. Case 1. The axes of T and U cross. Assume for example the attractive fixed point of U lies in $L(T)$ and hence the attractive fixed point of T lies in $R(U)$. If Φ is orientation preserving with respect to T then the attractive fixed point of $\Phi(U)$ lies to the left of $\alpha^*(\Phi(T))$ and hence the attractive fixed point of $\Phi(T)$ lies to the right of $\alpha^*(\Phi(U))$. That is Φ is orientation preserving with respect to U.

Case 2. The axes of T and U are disjoint yet T and U are not powers of boundary transformations. If we can find $V \in G$ whose axis crosses the axis of both T and U then we can apply Case 1 to obtain the desired result. To find V we again apply the simple argument of Tukia [10,1.4]. Take limit points ζ_1, ζ_2 of G such that the non-euclidean line joining them crosses the axis of both T and U. Find $V_i \in G$ with attractive fixed point v_i near ζ_i. For sufficiently large n, $V = V_1^n V_2^{-n}$ maps the exterior of a small circle about v_2 onto the interior of a small circle about v_1. Thus the fixed points of V will be close to ζ_1, ζ_2.

Case 3. $T = T_\gamma$ is a boundary transformation. Take a simple loop δ such that $\gamma\delta$ is a figure-8 loop. The direction of $\alpha^*(\Phi(T))$ can be deduced from that of $\alpha^*(\Phi(T_{\gamma\delta}))$.

2.9. The following lemma summarizes our results. We no longer need to assume all boundary transformations of G and H are hyperbolic. As above orient the axis of any hyperbolic transformation toward its attractive fixed point.

LEMMA 2.9. *Suppose* G *is finitely generated and* $\Phi: G \to H$ *is an isomorphism such that if* $B \in G$ *is a boundary transformation so is* $\Phi(B)$. *Then in fact* $\Phi(B)$ *is a boundary transformation if and only if*

B *is. Moreover the axes of* $S, T \in G$ *cross if and only if the axes of* $\Phi(S)$, $\Phi(T)$ *do. For* ***all*** *such* S, T *such that the attractive fixed point of* S *lies to the left of the axis of* T *one of the following occurs. Either the attractive fixed point of* $\Phi(S)$ *lies to the left of the axis of* $\Phi(T)$ *or it lies to the right. And if all fixed points of* $U \in G$ *lie to the left of the axis of* T *then in the former case all fixed points of* $\Phi(U)$ *lie to the left of the axis of* $\Phi(T)$ *and in the latter case, lie to the right.*

3. CONSTRUCTION OF THE HOMEOMORPHISM

3.1. In this chapter we will use Lemma 2.9 to piece together a homeomorphism (orientation reversing if Φ is) of $S(G) - b(G) \longrightarrow S(H) - b(H)$ that induces Φ. If T is hyperbolic we will write $\alpha^*(T)$ for the axis of T oriented toward the attractive fixed point of T and $\alpha(T)$ for the natural projection $\pi(\alpha^*(T))$ with orientation determined by $\alpha^*(T)$. However if T is a boundary transformation, or a power of one, $\Phi(T)$ may be parabolic (or conversely). It causes no problem and simplifies notation if we do not distinguish this case (after all, $\alpha(\Phi(T))$ may then be regarded as the limiting case of a simple loop).

If T is determined by a simple loop on $S(G) - b(G)$ then $\alpha(T)$ is a simple loop on $S(G) - b(G)$ except for what can be regarded as a limiting case that $\alpha(T)$ is a simple arc between two points of $b(G)$ of order two; we will not distinguish this case either. From the properties of geodesics (see §1.1) and from Lemma 2.9 we obtain the following result.

Property 7. $\alpha(T)$ is a simple loop in $S(G) - b(G)$ if and only if $\Phi(\alpha) \equiv \alpha(\Phi(T))$ is a simple loop in $S(H) - b(H)$. If $\alpha_1 = \alpha(T_1)$ and $\alpha_2 = \alpha(T_2)$ are simple loops then (a) $\alpha_1 \cap \alpha_2 = \phi$ if and only if $\Phi(\alpha_1) \cap \Phi(\alpha_2) = \phi$, and (b) α_1, α_2 cross at exactly one point if and only if $\Phi(\alpha_1)$, $\Phi(\alpha_2)$ do; furthermore if Φ is orientation preserving, α_2 crosses α_1 from left to right if and only if $\Phi(\alpha_2)$ so crosses $\Phi(\alpha_1)$ (the reverse is true if Φ is orientation reversing).

COROLLARY. genus $S(H)$ = genus $S(G)$, *number of ideal boundary components of* $S(H)$ = *corresponding number for* $S(G)$, *and* card $b(H)$ = card $b(G)$.

3.2. Case 1. The theorem is true if $S(G)$ is closed and $b(G) = \phi$.

Choose a set of $2g$ (g = genus $S(G)$) simple loops $\{\alpha_i\}$ on $S(G)$ such that α_i crosses α_{i+1} exactly once, $1 \leq i < 2g$, but otherwise $\alpha_i \cap \alpha_j = \phi$, $|i-j| \geq 2$. The α_i may be taken as geodesics. The result R_1 of cutting $S(G)$ along the α_i is simply connected. A fixed lift R_1^* of R_1 serves as a fundamental region for G in Δ: R_1^* has $8g-4$ sides arranged in pairs equivalent under G. There are $8g-4$ elements $T_j \in G$ such that the axis $\alpha^*(T_j)$ contains a side of R_1^*. By replacing T_j by T_j^{-1} if necessary we may assume R_1^* lies to the left of $\alpha^*(T_j)$.

It is clear by applying Property 7 that the geodesics $\beta_i = \Phi(\alpha_i)$ in $S(H)$ cut $S(H)$ into a simply connected region R_2. However it is easiest to follow the orientations by looking in Δ and applying Lemma 2.9. Doing that we see there is a lift R_2^* of R_2 bounded by one segment of each of the oriented axes $\beta_j^* = \alpha^*(\Phi(T_j))$ which fit together to give an orientation of ∂R_2^*. With respect to this particular orientation of ∂R_2^*, R_2^* lies either to the left (Φ orientation preserving) or to the right.

Consequently we can construct a homeomorphism $f^*: R_1^* \longrightarrow R_2^*$ which is consistent with the identifications of paired sides under G and Φ. f^* can be extended by the action of G, Φ, and H to map $\Delta \longrightarrow \Delta$

and is orientation reversing if and only if Φ is. The projection $f: S(G) \to S(H)$ is a homeomorphism.

3.3. Case 2. The theorem is true if $S(G) - b(G)$ is a triply connected plane region.

Assume first that all ideal boundary components of $S(G)$ and $S(H)$ are punctures. Then $S(G) - b(G)$ and $S(H) - b(H)$ are triply punctured spheres and the isomorphism Φ determines a one-to-one correspondence between the three punctures. We refer to Maskit [8, Lemma 7] for an elementary geometric proof that Φ is induced by a conformal or anti-conformal map between these punctured spheres. In fact Φ is of the form $T \to ATA^{-1}$ for some possibly orientation reversing Möbius transformation A.

Suppose now that some of the ideal boundary components of $S(G)$ and $S(H)$ are not punctures. Sew on to these once punctured disks. After doing this one obtains new groups G', H' and isomorphisms $I_1: G \to G'$, $I_2: H \to H'$ which are induced by (orientation preserving) homeomorphisms $f_i^*: \Delta \to \Delta$ obtained by lifting a homeomorphism $f_1: S(G) - b(G) \to S(G') - b(G')$ and similarly for H, $i = 1,2$. Furthermore the isomorphisms I_i preserve boundary transformations which in the case of G', H' are now parabolic. Hence from above we can find a homeomorphism $h^*: \Delta \to \Delta$ which induces the isomorphism $I_2 \Phi I_1^{-1}: G' \to H'$. The map $f_2^{*-1} h^* f_1^*: \Delta \to \Delta$ is what we are looking for.

3.4. Case 3. $S(G) - b(G)$ has one ideal boundary component.

Construct the system of geodesics α_i, $1 \leq i \leq 2g$, g = genus $S(G)$, on $S(G)$ and the corresponding system β_i, $1 \leq i \leq 2g$, on $S(H)$ as was done in Case 1. Recall from §1.1 that none of these geodesics pass through the points of $b(G)$ or $b(H)$. Then the results R_1, R_2 of cutting $S(G)$, $S(H)$ along these curves are doubly connected plane regions. Fix a lift R_1^* of R_1 and find as in Case 1 the corresponding lift R_2^* of R_2. The orientation of R_1 determines an orientation of the relative boundary $\partial_0 R_1^*$ of R_1^* (which is connected) so that R_1^* lies to its left. Now $\partial_0 R_1^*$ is composed of segments of a finite or infinite number of axes $\alpha^*(T)$ and we may choose T so that the orientation of $\alpha^*(T)$ agrees with that of $\partial_0 R_1^*$. Then in turn Φ determines an orientation of each segment of $\partial_0 R_2^*$ which by Lemma 2.9 is consistent from segment to segment giving an orientation to $\partial_0 R_2^*$; R_2^* lies to its left or right depending on whether Φ is orientation preserving or not.

Next R_1^* is preserved by a cyclic subgroup G_0 of G which also maps $\partial_0 R_1^*$ onto itself. In particular there is a connected fundamental set $\mathcal{F}_1 \subset \partial_0 R_1^*$ for the action of G_0 consisting of a finite union of sides of R_1^*. The negative end point p^* of $\mathcal{F}_1$ is mapped to the positive end $A(p^*)$ by a generator A of G_0. In R_1 draw a simple arc γ_1 from $p = \pi(p^*)$ to the point of $b(G)$, or to a point on the ideal boundary of $S(G)$ in its natural compactification. Let γ_1^* be the lift of γ from p^*. The arcs γ_1^* and $A(\gamma_1^*)$ bound a wedge σ_1 in R_1^* which serves as a fundamental region for the action of G_0 in R_1^*.

Likewise there is a fundamental set $\mathcal{F}_2 \subset \partial_0 R_2^*$ for the action of $\Phi(G_0)$ consisting of the corresponding sides of R_2^*. Automatically $\Phi(A)$ maps the negative end q^* of $\mathcal{F}_2$ (in the orientation of $\partial_0 R_2^*$) to the positive end. And there is a corresponding wedge σ_2 serving as a fundamental region for $\Phi(G_0)$ in R_2^*. Consequently we can construct a homeomorphism $f^*: \sigma_1 \longrightarrow \sigma_2$ that on the boundary sends points equivalent under G to points equivalent under H. f^* extends to a homeomorphism $\Delta \longrightarrow \Delta$ which is what is needed.

3.5. We can now complete the proof of the theorem. Assume $S(G)$ does not fall into one of the previous categories. Draw mutually disjoint simple loops $\gamma_1, \ldots, \gamma_n$ on $S(G) - b(G)$ such that (*i*) γ_1 is the relative boundary of a subregion $R_1 \subset S(G) - b(G)$ which is compact of genus equal to the genus of $S(G)$ if this is positive, otherwise is a triply connected region.

(*ii*) γ_i and γ_{i+1}, $1 \leq i \leq n-1$, are the relative boundary components in $S(G) - b(G)$ of a triply connected subregion R_{i+1},

(*iii*) γ_i separates γ_{i-1} from γ_{i+1},

(*iv*) γ_n bounds a triply connected subregion $R_{n+1} \subset S(G) - b(G)$.

We may assume each γ_i is a geodesic (see §1.1) and assume first that γ_1 and γ_n are non-degenerate (the others are always). Fix a lift R_i^* of R_i for each i so that R_{i+1}^* is adjacent to R_i^* along an axis lying over γ_i. Let G_i be the subgroup of G that preserves R_i^*. Thus the relative boundary $\partial_0 R_i^*$ of R_i^* in Δ consists of an axis over γ_i, one over γ_{i+1}, and their orbit under the infinite group G_i.

There is a region S_i^* corresponding to R_i^* in that the axis $\alpha^*(T)$ is contained in $\partial_0 R_i^*$ if and only if $\alpha^*(\Phi(T)) \subset \partial_0 S_i^*$ (Lemma 2.9). S_{i+1}^* is adjacent to S_i^*. Because $\partial_0 S_i^*$ is preserved by $\Phi(G_i) = H_i$, S_i^* is too. An element $T \in G_i$ is a boundary transformation with respect to G_i if it is also a boundary transformation with respect to G or if its axis lies in $\partial_0 R_i^*$ and similarly for the boundary transformations of H_i. Consequently the isomorphism $\Phi\colon G_i \longrightarrow H_i$ preserves boundary transformations.

Using Case 2 or 3 we can construct a homeomorphism $f_1^*\colon R_1^* \longrightarrow S_1^*$ which induces $\Phi\colon G_1 \longrightarrow H_1$. Using Case 2 there is a homeomorphism $f_2^*\colon R_2^* \longrightarrow S_2^*$ which induces $\Phi\colon G_2 \longrightarrow H_2$. Let $\alpha^*(T)$ be the axis that separates R_1^* from R_2^*. We must match f_1^*, f_2^* across $\alpha^*(T)$.

Project f_1^*, f_2^* to obtain homeomorphisms $f_1\colon R_1 \longrightarrow S_1$, $f_2\colon R_2 \longrightarrow S_2$ where R_1 and R_2 are adjacent along the geodesic $\alpha(T)$ while S_1 and S_2 are adjacent along $\alpha(\Phi(T))$. Fix a closed annular neighborhood $K \subset R_2 - R_2 \cap b(G)$ of $\alpha(T)$ and denote by K^* the lift of K adjacent to $\alpha^*(T)$. The map f_1 on $\alpha(T) \subset \partial K$ and f_2 restricted to the other component β of ∂K give homeomorphisms of the components of ∂K to those of $\partial f_2(K)$. Extend these boundary maps to (a) a homeomorphism $h: K \to f_2(K)$ with the restriction (b) the homotopy class of h with property (a) is chosen so that the lift h^* of h to $K^* \to f_2^*(K^*)$ that agrees with f_1^* on $\alpha^*(T)$ also agrees with f_2^* on the part of ∂K^* over β. Define $g = f_1$ on R_1, $g = h$ on K, and $g = f_2$ on $R_2 - K$. Let g^* be the lift of g to $R_1^* \cup R_2^*$ that agrees with f_1^* on R_1^* and f_2^* on $R_2^* - K$.

In this manner proceed from R_i^* to R_{i+1}^* ending at R_n^* with a homeomorphism $g^*: \cup R_i^* \to \cup S_i^*$. Finally use the action of G and H to extend f^* to all Δ.

In the exceptional case that γ_n, for instance, is degenerate then γ_n is an arc between two points of order two in $b(G)$. We can regard $R_n = \gamma_n$. Then S_n too reduces to an arc and the required map $R_n \to S_n$ is clear.

4. THE INTERSECTING AXES THEOREM

4.1. The first statement of the following proposition is due to Fenchel-Nielsen [2]. There is another unpublished proof due to L. Keen and B. Maskit (personal communication) which uses combination theorems. A proof of a special case appears in Maskit's paper [8]. Tukia [10] independently rediscovered the result with some restrictions. The second statement follows from [1] or [6].

PROPOSITION 4.1. *Suppose there is an isomorphism* $\Phi: G \to H$ *between fuchsian groups with the property that* $T_1, T_2 \in G$ *have intersecting axes if and only if* $\Phi(T_1)$ *and* $\Phi(T_2)$ *also do. Then there is a homeomorphism* $f: \Delta \to \Delta$ *which induces* Φ. *The homotopy type of* f *in* $S(G) - b(G)$ *is uniquely determined.*

4.2. An element $T = T_\gamma$ of G which is determined by a simple loop $\gamma \subset S(G) - b(G)$ is characterized as a boundary transformation either by its being parabolic, or by the property that $\alpha(T) \cap \alpha(S) = \phi$ for all $S \in G$ not a power of T. For this reason when G is finitely generated the proposition is a special case of the theorem.

4.3. To prove the proposition when G is not finitely generated recall that $b(G)$ is a discrete set in $S(G)$. Select an exhaustion $\{\Omega_n\}$ of $S(G)$ with $\Omega_n \subset \Omega_{n+1}$ and $\partial\Omega_n \subset S(G) - b(G)$ satisfying the following properties.

(a) $\Omega_n - b(G) \cap \Omega_n$ has finite Euler characteristic.

(b) Each component of $\partial\Omega_n$ is compact in $S(G) - b(G)$ and bounds a component of $S(G) - b(G) - \Omega_n$ of infinite Euler characteristic.

(c) $\cup\, \Omega_n = S(G)$.

We can also assume each component of $\partial\Omega_n$ is a geodesic.

Fix lifts $\{\Omega_n^*\}$ of $\{\Omega_n\}$ so that $\Omega_n^* \subset \Omega_{n+1}^*$. The relative $\partial_0\Omega_n^*$ in Δ is a union of axes lying over $\partial\Omega_n$ and $\lim \Omega_n^* = \Delta$. Let G_n be the finitely generated subgroup of G that preserves Ω_n^*; then $G_n \subset G_{n+1}$. Moreover $S(G_n)$ contains Ω_n and in fact $S(G_n) - \Omega_n$ is a union of annuli containing no points of $b(G_n)$.

Start by applying the theorem to G_1 and $H_1 = \Phi(G_1)$. A boundary transformation of G_1 is either a boundary transformation of G or its axis is contained in $\partial_0\Omega_1^*$. The hypothesis on Φ implies that Φ sends these to boundary transformations in H_1. Thus there is a homeomorphism $f_1^*\colon \Delta \longrightarrow \Delta$ which induces $\Phi\colon G_1 \longrightarrow H_1$. For definiteness, assume f_1^* is orientation preserving. Let $\Omega_1'^*$ be the region determined by the property that $\alpha^*(T) \subset \partial_0\Omega_1^*$ if and only if $\alpha^*(\Phi(T)) \subset \partial_0\Omega_1'^*$. Then $S(H_1) - \Omega_1'$ is a union of annuli containing no points of $b(H_1)$. Hence we may assume that $f_1^*: \Omega_1^* \to \Omega_1'^*$.

We will proceed by induction. Assume $f_{n-1}^*: \Omega_{n-1}^* \to \Omega_{n-1}'^*$ is an orientation preserving homeomorphism which induces an isomorphism $\Phi: G_{n-1} \to \Phi(G_{n-1}) = H_{n-1}$. Here $\Omega_n'^*$ is determined from Ω_n^* as $\Omega_1'^*$ is from Ω_1^*. We will show how to extend f_{n-1}^* to $f_n^*: \Omega_n^* \to \Omega_n'^*$.

Let Δ_n be a component of $\Omega_n^* - \Omega_{n-1}^*$ and let $G_n' \subset G_n$ be the subgroup that preserves it. So Δ_n lies over a component of $\Omega_n - \Omega_{n-1}$. Because $\Phi: G_n' \to H_n' = \Phi(G_n')$ has the intersecting axis property there is a homeomorphism $g^*: \Delta \to \Delta$ that induces it. To show that g^* is orientation preserving we have to know whether Φ preserves the relative orientation of the axes of G_n'. This is most easily seen by applying Lemma 2.9 to the full group G_n. Recall that the behavior under Φ of merely one pair of intersecting axes (in this case in $G_1 \subset G_n$) determines the orientation. Now $S(G_n') - \pi(\Delta_n^*)$ is a union of annuli free of points of $b(G_n')$ and likewise for the corresponding $S(H_n') - \pi(\Delta_n'^*)$. Thus we can also assume $g^*: \Delta_n^* \to \Delta_n'^*$. Carry this process out for one lift of each component of $\Omega_n - \Omega_{n-1}$.

The regions $\Delta_n'^*$ are situated adjacent to $\Omega_{n-1}'^*$ exactly as the Δ_n^* are situated adjacent to Ω_{n-1}^* (i.e., corresponding axes under Φ are involved). Therefore the result of adjoining the orbit of the $\Delta_n'^*$ under H_{n-1} to $\Omega_n'^*$ gives a region $\Omega_n'^*$ corresponding to Ω_n^*. We have the map $f_{n-1}^*: \Omega_{n-1}^* \to \Omega_{n-1}'^*$ and $g^*: \Omega_n^* - \Omega_{n-1}^* \to \Omega_n'^* - \Omega_{n-1}'^*$. It remains only to patch these maps together across the common boundary axes. But this is carried out as in §3.5, giving f_n^*.

4.4. The following is an application of the theorem and Lemma 2.9 to Teichmüller theory.

COROLLARY. *Assume* $\Phi: G \to H$ *is an isomorphism between finitely generated fuchsian groups such that* Φ *gives a one-to-one correspondence between boundary transformations in which parabolic transformations* *correspond to parabolic transformations. Then* G *and* H *belong to the same Teichmüller space.*

4.5. We close by citing some related results in a different direction. Sorvali [9] has made an interesting study of isomorphisms $\Phi: G \rightarrow H$ in terms of their effect on multipliers and cross ratios of fixed points of G. If in addition to the intersecting axes property an isomorphism $\Phi: G \rightarrow H$ preserves parabolic elements then the resulting homeomorphism $f^*: \Delta \rightarrow \Delta$ that induces Φ can be extended to a homeomorphism $h^*: \partial\Delta \rightarrow \partial\Delta$ which induces Φ there. f^* is determined by Φ only up to homotopy but h^* is uniquely determined (if G is of the first kind). Lehto [3] has raised the question whether if h^* is quasisymmetric f^* can be chosen to be quasiconformal (this is non-trivial if G is not finitely generated) and answered the question affirmatively for groups of the first kind without torsion when the distortion caused by h^* is not too great. These results point us in the direction of the ergodic properties of the boundary mappings which is a deep and beautiful subject but one we cannot deal with here.

REFERENCES

1. L. Bers and L. Greenberg, *Isomorphisms between Teichmüller spaces*, in Advances in the Theorey of Riemann Surfaces, Ann. of Math. Studies, vol. 66, Princeton Univ. Press, 1970.

2. W. Fenchel and J. Nielsen, Discontinuous Groups of non-Euclidean Motions, to appear.

3. O. Lehto, *Group isomorphisms induced by quasiconformal mappings*, in Contributions to Analysis, Academic Press, N.Y., 1974.

4. A.M. Macbeath, *Geometrical realisations of isomorphisms between plane groups*, Bull. Amer. Math. Soc. 71(1965), 629-630.

5. A. Marden, *On finitely generated Fuchsian groups*, Comm. Math. Helv. 42(1967), 81-85.

6. A. Marden, *On homotopic mappings of Riemann surfaces*, Ann. of Math. 90(1969), 1-8.

7. A. Marden, I. Richards and B. Rodin, *Analytic self-mappings of Riemann surfaces*, Jour. d'Anal. Math. 18(1967), 197-225.

8. B. Maskit, *On boundaries of Teichmüller spaces II*, Ann. of Math. 91(1970), 607-639.

9. T. Sorvali, *The boundary mapping induced by an isomorphism of covering groups*, Ann. Acad. Sci. Fenn. AI 526(1972).

10. P. Tukia, *On discrete groups of the unit disk and their isomorphisms*, Ann. Acad. Sci. Fenn. AI 504(1972).

11. F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. 87(1968), 56-88.

12. H. Zieschang, *Über Automorphismen ebener Gruppen*, Dokl. Akad. Nauk SSSR 155(1964), 57-60.

13. H. Zieschang, *Über Automorphismen ebener diskontinuierlicher Gruppen*, Math. Ann. 166(1966), 148-167.

14. H. Zieschang, E. Vogt, H.D. Coldewey, *Flächen und ebene diskontinuierliche Gruppen*, Lecture Notes in Math. no 122, Springer-Verlag, New York, 1970.